Th$^m$ 17.5

Second Derivative Test for functions of 2 variables

Let $f(x,y)$ be a function with cont$^c$ partial derivatives of the first & second order in a domain S, & let $(x_0, y_0)$ be an interior pt. of S that is a stationary pt. for f.

Let $A = f''_{11}(x_0,y_0)$, $B = f''_{12}(x_0,y_0)$ & $C = f''_{22}(x_0,y_0)$

(a) If $A<0$ & $AC-B^2>0$, then $(x_0,y_0)$ is a local maxm pt.

(b) If $A>0$ & $AC-B^2>0$, then $(x_0,y_0)$ is a local minm pt.

(c) If $AC-B^2<0$, then $(x_0,y_0)$ is a saddle pt.

(d) If $AC-B^2=0$, then $(x_0,y_0)$ could be a local maxm, a local minm or a saddle pt.

Q. Find all the stationary pts. of $f(x,y) = x^4+2y^2-2xy$ & classify them as local max., local min or saddle point.

$$f'_1(x,y) = 4x^3-2y = 0 \Rightarrow 2x^3-y=0$$

$$f'_2(x,y) = 4y-2x=0 \Rightarrow 2y-x=0 \Rightarrow \boxed{x=2y}$$

$$2(8y^3)-y=0$$
$$16y^3-y=0$$
$$y(16y^2-1)=0$$
$$y=0,\quad y^2=1/16$$
$$y=0,\quad y=\pm 1/4$$

Stationary pts. of f: $(0,0)$, $\left(\frac{1}{2},\frac{1}{4}\right)$, $\left(\frac{-1}{2},\frac{-1}{4}\right)$

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$f_1' = 4x^3 - 2y$ | $f_2' = 4y - 2x$

$A = f_{11}'' = 12x^2$ | $C = f_{22}'' = 4$

$B = f_{12}'' = -2$

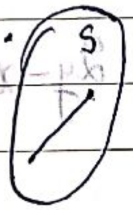
| Point | A | B | C | $AC - B^2$ | Type |
|---|---|---|---|---|---|
| $(0,0)$ | $0$ | $-2$ | $4$ | $-4$ | Saddle Point |
| $\left(\frac{1}{2}, \frac{1}{4}\right)$ | $12\left(\frac{1}{4}\right) = 3$ | $-2$ | $4$ | $12 - 4 = 8$ | Local minm |
| $\left(-\frac{1}{2}, -\frac{1}{4}\right)$ | $3$ | $-2$ | $4$ | $8$ | Local minm |

<u>Convex Sets</u>

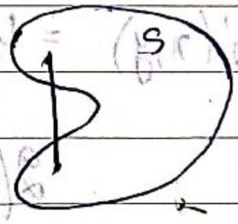
<u>defn</u> A set $S$ in $R^n$ is convex if each pair of pts. in $S$ can be joined by a line segment lying entirely within $S$.

$x \in S$, $y \in S$ & $\lambda \in [0,1] \Rightarrow (1-\lambda)x + \lambda y \in S$

eg. $R^2$




$\boxed{\text{convex}}$




$\boxed{\text{not convex}}$

eg. Let $U(x) = U(x_1, \ldots, x_n)$

If $U(x^0) = a$, then the <u>upper-level set</u>

$\Gamma_a = \{x : U(x) \geq a\}$ consists of all bundles that give at least as much util as bundle $x^0$.

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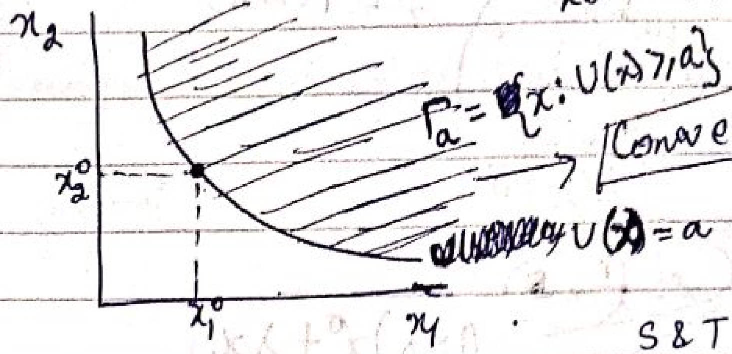
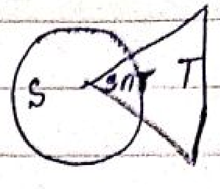
$U(x) = U(x_1, x_2)$  Let $U(x^0) = a$



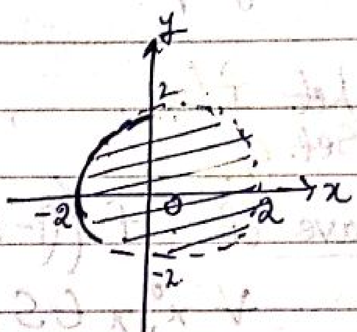




S & T are convex sets in $R^n \Rightarrow S \cap T$ is convex

**Result** $S_1, \dots, S_m$ are convex in $R^n \Rightarrow S_1 \cap \dots \cap S_m$ is convex

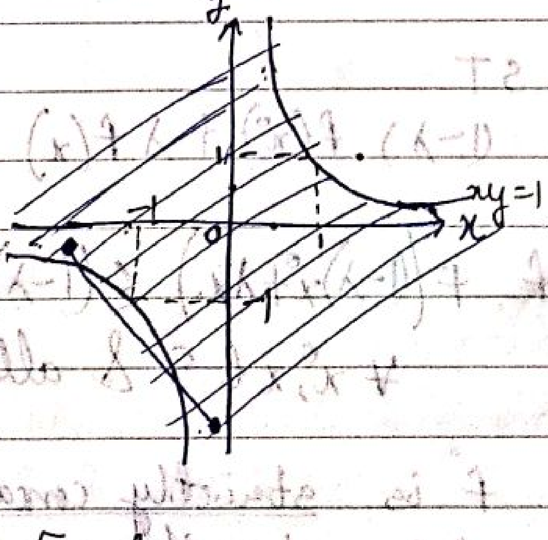
Q- ~~Decide~~ Which of the foll. sets are convex?

(a) $\{(x,y) : x^2 + y^2 < 4\}$




Convex

(b) $\{(x,y) : xy \leq 1\}$




Not Convex

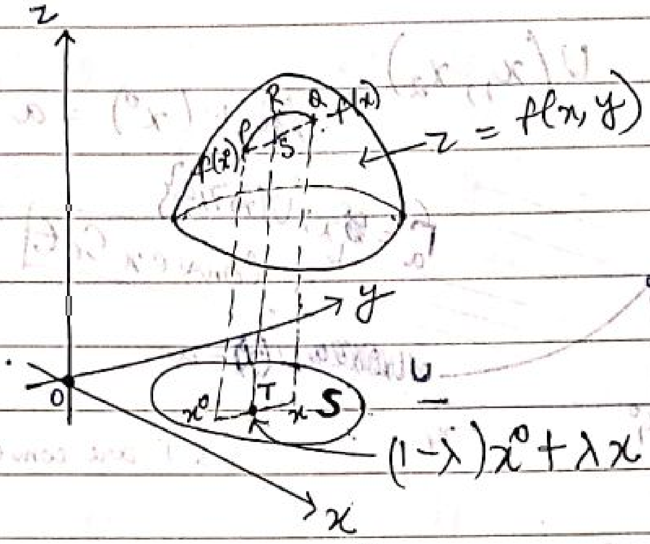
**Concave & Convex Functions**

Let $f(x_1, \dots, x_n)$ ~~$f(x,y)$~~ be defined on a convex set $S$ in $R^n$.

$\boxed{\text{Geometric definition}}$

① $f$(~~$x,y$~~) is concave if the line segment joining any 2 pts on the graph is never above the graph.

eg.




(2) f(~~x, y~~) is convex if the line segment joining any 2 pts. on the graph is never below the graph.

$\boxed{\text{Algebraic definition}}$ Let $f(x_1, \_\_\_\_, x_n)$ be defined on a convex set S.

(1) ~~A function~~ f is concave if $f((1-\lambda)x^0 + \lambda x) \geq (1-\lambda) f(x^0) + \lambda f(x)$
$\forall\, x^0, x \in S$ & all $\lambda \in (0,1)$

$$RT \geq ST$$
$$f((1-\lambda)x^0 + \lambda x) \geq (1-\lambda) f(x^0) + \lambda f(x)$$

(2) f is convex if $f((1-\lambda)x^0 + \lambda x) \leq (1-\lambda) f(x^0) + \lambda f(x)$
$\forall\, x^0, x \in S$ & all $\lambda \in (0,1)$

If $\geq$ is replaced by $>$, f is strictly concave

If $\leq$ is replaced by $<$, f is strictly convex

Results on Concave & Convex functions

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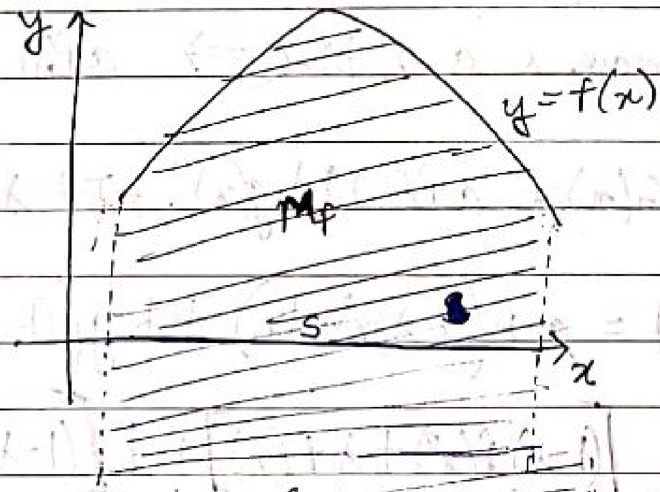
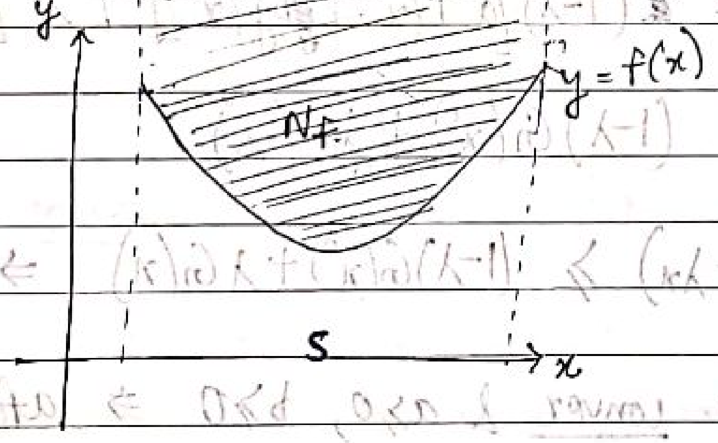
( One Variable)

① (a) $f$ is concave $\iff$ $M_f = \{(x,y) : x \in S \ \& \ y \le f(x)\}$ is convex.

$M_f$ denotes the set of all pts. on or below the graph of $f$.




(b) $f$ is convex $\iff$ $N_f = \{(x,y) : x \in S \ \& \ y \ge f(x)\}$ is convex.




( n variables)

② Jensen's Inequality

Let $f(x_1, \ldots, x_n)$ be defined on a convex set $S$ in $R^n$:

(a) $f$ is concave iff the foll. inequality is satisfied for all $x_1, \ldots, x_m$ in $S$ & all $\lambda_1 \ge 0, \ldots, \lambda_m \ge 0$ with $\lambda_1 + \ldots + \lambda_m = 1$:

$$f(\lambda_1 \underline{x_1} + \ldots + \lambda_m \underline{x_m}) \geq \lambda_1 f(x_1) + \ldots + \lambda_m f(x_m)$$

(b) $f$ is convex

$$\le$$

Nonnegative real no.s that sum to one are called convex weights.

## Useful Conditions for Concavity & Convexity

**Theorem 17.6**

Let $f(x)$ & $g(x)$ be defined over a <u>convex set</u> $S$ in $R^n$. Then:

(a) $f(x)$ & $g(x)$ are <u>concave</u> & $a \geq 0, b \geq 0 \Rightarrow af(x) + bg(x)$ is <u>concave</u>

<u>Proof</u>: Let $G(x) = af(x) + bg(x)$. If $\lambda \in (0,1)$ & $x^0, x \in S$, then:

$$G((1-\lambda)x^0 + \lambda x) = af((1-\lambda)x^0 + \lambda x) + bg((1-\lambda)x^0 + \lambda x)$$

$$\geq a[(1-\lambda)f(x^0) + \lambda f(x)] + b[(1-\lambda)g(x^0) + \lambda g(x)]$$

(using concavity of $f$ & $g$)

$$= (1-\lambda)[af(x^0) + bg(x^0)] + \lambda[af(x) + bg(x)]$$

$$= (1-\lambda)G(x^0) + \lambda G(x)$$

$$G((1-\lambda)x^0 + \lambda x) \geq (1-\lambda)G(x^0) + \lambda G(x) \Rightarrow G \text{ is concave.}$$

(b) $f(x)$ & $g(x)$ are <u>convex</u> & $a \geq 0, b \geq 0 \Rightarrow af(x) + bg(x)$ is <u>convex</u>

(c) $f(x)$ is <u>concave</u> & $F(u)$ is <u>concave & ↑ing</u> $\Rightarrow U(x) = F(f(x))$ is <u>concave</u>

<u>Proof</u> Let $\lambda \in (0,1)$ & $x, x^0 \in S$. Then:

$$U((1-\lambda)x^0 + \lambda x) = F(f((1-\lambda)x^0 + \lambda x))$$

$$\geq F((1-\lambda)f(x^0) + \lambda f(x)) \quad \text{(concavity of } f \text{ \& } F \text{ is ↑ing)}$$

$$\geq (1-\lambda)F(f(x^0)) + \lambda F(f(x)) \quad \text{(concavity of } F\text{)}$$

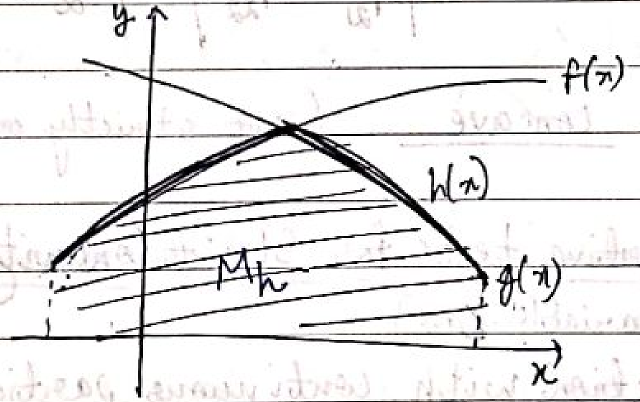
$$= (1-\lambda)U(x^0) + \lambda U(x)$$

$\Rightarrow U\left((1-\lambda)x^0+\lambda x\right) \geq (1-\lambda)U(x^0)+\lambda U(x)$

$\Rightarrow U(x)$ is concave.

(d) $f(x)$ is convex & $F(u)$ is convex & $\uparrow$ ing $\Rightarrow U(x)=F(f(x))$ is convex

(e) $f(x)$ & $g(x)$ are concave $\Rightarrow h(x) = \min\{f(x), g(x)\}$ is concave




$\underbrace{M_f}_{\text{convex}} \cap \underbrace{M_g}_{\text{convex}} = \underbrace{M_h}_{\text{convex}}$

$M_h$ is convex $\Rightarrow h(x)$ is concave.

(f) $f(x)$ & $g(x)$ are convex $\Rightarrow h(x) = \max.\{f(x), g(x)\}$ is convex

**Theorem 17.9** Second Derivative Tests for Concavity / Convexity (2 variable case)

Let $z=f(x,y)$ be a function with continuous partial derivatives of the first & second order, defined over an open convex set S in the plane. Then:

(a) $f$ is concave $\Leftrightarrow f''_{11} \leq 0,\ f''_{22} \leq 0$ & $\begin{vmatrix} f''_{11} & f''_{12} \\ f''_{21} & f''_{22} \end{vmatrix} \geq 0$

(b) $f$ is convex $\Leftrightarrow f''_{11} \geq 0,\ f''_{22} \geq 0$ & $\begin{vmatrix} f''_{11} & f''_{12} \\ f''_{21} & f''_{22} \end{vmatrix} \geq 0$

Q- Let $f(x,y) = 2x - y - x^2 + 2xy - y^2$. Is $f$ concave/convex?

$$f_1' = 2 - 2x + 2y, \quad f_{11}'' = -2, \quad f_{12}'' = 2$$

$$f_2' = -1 + 2x - 2y, \quad f_{22}'' = -2$$

$$f_{11}'' \leq 0, \quad f_{22}'' \leq 0, \quad \begin{vmatrix} f_{11}'' & f_{12}'' \\ f_{21}'' & f_{22}'' \end{vmatrix} = \begin{vmatrix} -2 & 2 \\ 2 & -2 \end{vmatrix} = 0 \geq 0$$

$\Rightarrow f$ is concave (not strictly concave)

17.10

**Theorem:** Second-derivative tests for Strict Concavity / Strict Convexity (2-variable case)

Let $z = f(x,y)$ be a function with continuous partial derivatives of the first & second order defined over an open convex set $S$ in the plane. Then:

(a) $f_{11}'' < 0$ & $\begin{vmatrix} f_{11}'' & f_{12}'' \\ f_{21}'' & f_{22}'' \end{vmatrix} > 0 \Rightarrow f$ is strictly concave.

(b) $f_{11}'' > 0$ & $\begin{vmatrix} f_{11}'' & f_{12}'' \\ f_{21}'' & f_{22}'' \end{vmatrix} > 0 \Rightarrow f$ is strictly convex.

Q- Let $f(x,y) = x + y - e^x - e^{x+y}$. Is $f$ strictly concave / strictly convex?

$$f_1' = 1 - e^x - e^{x+y}, \quad f_{11}'' = -e^x - e^{x+y} < 0, \quad f_{12}'' = -e^{x+y}$$

$$f_2' = 1 - e^{x+y}, \quad f_{22}'' = -e^{x+y}$$

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$$\begin{vmatrix} f_{11}'' & f_{12}'' \\ f_{21}'' & f_{22}'' \end{vmatrix} = (-e^x - e^{x+y})(1 - e^{x+y}) - (e^{x+y})^2$$

$$= e^{2x+y} + \cancel{(e^{x+y})^2} - \cancel{(e^{x+y})^2}$$

$$= e^{2x+y} > 0$$

$\Rightarrow$ f is ~~strictly concave.~~ (Concave)

17.7

**Theorem** Suppose that $f(x) = f(x_1, \_\_\_, x_n)$ has continuous partial derivatives in an open, convex set $S$ in $R^n$. Then:

(a) f is concave iff for all $x^0, x \in S$

$$f(x) - f(x^0) \leq \frac{\partial f(x^0)}{\partial x_1}(x_1 - x_1^0) + \_\_\_ + \frac{\partial f(x^0)}{\partial x_n}(x_n - x_n^0)$$

$$f(x) - f(x^0) \leq \sum_{i=1}^{n} \frac{\partial f(x^0)}{\partial x_i}(x_i - x_i^0)$$

(b) f is strictly concave iff $\leq$ is replaced by $<$ in part(a).

(c) ~~The corresponding~~ f is convex iff $\leq$ is replaced by $\geq$ in (a).

(d) f is strictly convex iff $\leq$ is replaced by $>$ in (a).

17.8 Concavity/Convexity & Global Max/Min.

**Theorem** Suppose that $f(x)$ has continuous partial derivatives in a convex set $S$ in $R^n$, & let $x^0$ be an interior pt. in $S$. Then.

(a) If f is $\boxed{\text{concave}}$, then:

$x^0$ is a stationary pt. of f $\Leftrightarrow$ $x^0$ is a (global) maxm pt. of f in S

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**Proof** $x^0$ is a stationary pt. of $f \Rightarrow x^0$ is a (global) maxm pt. of $f$

$$\frac{\partial f(x^0)}{\partial x_i} = 0 \text{ for } i = 1, \ldots, n$$

$$\Rightarrow f(x) - f(x^0) \leq 0 \quad (\text{Th}^{m} \; 17.7)$$

$$\Rightarrow f(x) \leq f(x^0) \quad \forall x \in S$$

$$\Rightarrow x^0 \text{ is a (global) max}^m \text{ pt. of } f.$$

(b) If $f$ is $\boxed{\text{convex}}$, then:

$x^0$ is a stationary pt. of $f \Leftrightarrow x^0$ is a (global) minm pt. of $f$.

**Proof** $x^0$ is a stationary pt. of $f$.

$$\frac{\partial f(x^0)}{\partial x_i} = 0 \text{ for } i = 1, \ldots, n$$

$$f(x) - f(x_0) \geq 0 \quad (\text{Th}^{m} \; 4)$$

$$\Rightarrow f(x) \geq f(x_0) \quad \forall x \in S$$

$$\Rightarrow x_0 \text{ is a (global) min}^m \text{ pt. of } f.$$

($\text{Th}^{m}$ 17.9 + $\text{Th}^{m}$ 17.8)

**Theorem 17.11** Sufficient Conditions for Global Extreme Points (2-variable case)

Let $f(x, y)$ be a function with continuous partial derivatives of the first & second order defined over a convex set $S$ & let $(x_0, y_0)$ be an interior pt. of $S$ at which $f$ is stationary.

(a) If for all $(x, y)$ in $S$, $f_{11}'' \leq 0$, $f_{22}'' \leq 0$ & $\begin{vmatrix} f_{11}'' & f_{12}'' \\ f_{21}'' & f_{22}'' \end{vmatrix} \geq 0$,

then $(x_0, y_0)$ is a (global) maxm pt. of $f$ in $S$.

Proof: For all $(x,y)$ in $S$, $f_{11}'' \le 0$, $f_{22}'' \le 0$ & $\begin{vmatrix} f_{11}'' & f_{12}'' \\ f_{21}'' & f_{22}'' \end{vmatrix} \ge 0 \xrightarrow{Th^m\ 17.9} f$ is concave

$\Big\downarrow$ Th$^m$ 17.8

$(x_0, y_0)$ is a (global) max$^m$ pt. of $f$.

(b) If, for all $(x,y)$ in $S$, $f_{11}'' \ge 0$, $f_{22}'' \ge 0$ & $\begin{vmatrix} f_{11}'' & f_{12}'' \\ f_{21}'' & f_{22}'' \end{vmatrix} \ge 0$, then $(x_0, y_0)$ is a (global) minim pt. of $f$ in $S$.

Proof: For all $(x,y)$ in $S$, $f_{11}'' \ge 0$, $f_{22}'' \ge 0$ & $\begin{vmatrix} f_{11}'' & f_{12}'' \\ f_{21}'' & f_{22}'' \end{vmatrix} \ge 0$

$\Big\downarrow$ Th$^m$ 17.9

$(x_0, y_0)$ is a (global) minim pt. of $f$. $\xleftarrow{Th^m\ 17.8}$ $f$ is convex

Q- Show that the function $f$ defined by
$$f(x,y) = -2x^2 - y^2 + 4x + 4y - 3$$
for all $(x,y)$ has a (global) max$^m$ at $(x,y) = (1,2)$.

$f_1' = -4x + 4$, $f_{11}'' = -4$, $f_{12}'' = 0$

$f_2' = -2y + 4$, $f_{22}'' = -2$

$f_1' = f_2' = 0$ at $(1,2)$ $\Rightarrow$ $(1,2)$ is a stationary pt. of $f$

$f_{11}'' \le 0$, $f_{22}'' \le 0$ & $\begin{vmatrix} -4 & 0 \\ 0 & -2 \end{vmatrix} = 8 \ge 0 \Rightarrow (1,2)$ is a (global) max$^m$ pt. of $f$

(Th$^m$ 17.11 (a))

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section 17.9

( 3 variables only)

( Theorem 3')

## Second-Derivative Test for Strict Concavity / Strict Convexity (n-variable case)

Let $z = f(x) = f(x_1, \ldots, x_n)$ be a $C^2$ function i.e. it has continuous partial derivatives of the first & second order, defined over a convex set $S$ in $R^n$.

Hessian matrix of $f$ at $x$.

$$H(x) = \begin{array}{c} \begin{array}{ccccc} D_1 & D_2 & D_3 & & \end{array} \\ \begin{pmatrix} f''_{11} & f''_{12} & f''_{13} & \cdots & f''_{1n} \\ f''_{21} & f''_{22} & f''_{23} & \cdots & f''_{2n} \\ f''_{31} & f''_{32} & f''_{33} & \cdots & f''_{3n} \\ \vdots & \vdots & & & \vdots \\ f''_{n1} & f''_{n2} & \cdots & \cdots & f''_{nn} \end{pmatrix} \end{array}$$

Defn The $k^{th}$-order leading principal minor of $H$, $D_k(x)$, is obtained by deleting the last $n-k$ rows & the last $n-k$ columns from $H$.

eg. first order LPM $= f''_{11}$ $\quad D_1(x)$

$D_2(x)$ second order LPM $= \begin{vmatrix} f''_{11} & f''_{12} \\ f''_{21} & f''_{22} \end{vmatrix}$

$D_3(x)$ third order LPM $= \begin{vmatrix} f''_{11} & f''_{12} & f''_{13} \\ f''_{21} & f''_{22} & f''_{23} \\ f''_{31} & f''_{32} & f''_{33} \end{vmatrix}$

$\vdots$

$$D_k(x) = \begin{vmatrix} f''_{11} & f''_{12} & \cdots & f''_{1k} \\ f''_{21} & f''_{22} & \cdots & f''_{2k} \\ \vdots & \vdots & & \vdots \\ f''_{k1} & f''_{k2} & \cdots & f''_{kk} \end{vmatrix} \qquad (k = 1, 2, \ldots, n)$$

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1(a) $\boxed{(-1)^k D_k(x) > 0}$ for $k = 1, \ldots, n$ & $\forall\, x \in S \Rightarrow f$ is strictly concave

$D_1(x) < 0,\ D_2(x) > 0,\ D_3(x) < 0,\ D_4(x) > 0 \ldots$

(b) $\boxed{D_k(x) > 0}$ for $k = 1, \ldots, n$ & $\forall\, x \in S \Rightarrow f$ is strictly convex.

Second derivative test for Concavity / Convexity (n-variable case)

(Theorem 2')

Defn: $r$-th order Principal Minor $\Delta_r(x)$ of $H$ is obtained by deleting any $(n-r)$ rows & the same $(n-r)$ columns of $H$.

eg. $H(x) = \begin{pmatrix} f''_{11} & f''_{12} & f''_{13} \\ f''_{21} & f''_{22} & f''_{23} \\ f''_{31} & f''_{32} & f''_{33} \end{pmatrix}$

$\Delta_1(x)$: $f''_{11},\ f''_{22},\ f''_{33}$

$\Delta_2(x)$: $\begin{vmatrix} f''_{11} & f''_{12} \\ f''_{21} & f''_{22} \end{vmatrix},\ \begin{vmatrix} f''_{22} & f''_{23} \\ f''_{32} & f''_{33} \end{vmatrix},\ \begin{vmatrix} f''_{11} & f''_{13} \\ f''_{31} & f''_{33} \end{vmatrix}$

(a) $f$ is concave $\Leftrightarrow \boxed{(-1)^r \Delta_r(x) \geq 0}$ for all principal minors of order $r = 1, \ldots, n$ & for all $x \in S$.

(b) $f$ is convex $\Leftrightarrow \boxed{\Delta_r(x) \geq 0}$ for all principal minors of order $r = 1, \ldots, n$ & for all $x \in S$.

Definiteness & Strict Concavity/Strict Convexity.

~~Definiteness~~ Thm 17.1 Let $f(x) = f(x_1, x_2, x_3, \ldots x_n)$ be a function with continuous partial derivatives of the first & second order, defined over a convex set $S$ in $R^n$.

(a) $(-1)^k D_k(x) > 0$ for $k = 1, 2, 3, \ldots, n$ & $\forall\, x \in S \Rightarrow f$ is strictly concave

$\Updownarrow$

$\boxed{H(x) \text{ is Negative definite } \forall\, x \in S \Rightarrow f \text{ is strictly concave}}$

(b) $D_k(x) > 0$ for $k = 1, 2, 3, \ldots, n$ & $\forall\, x \in S \Rightarrow f$ is strictly convex

$\Updownarrow$

$\boxed{H(x) \text{ is Positive definite } \forall\, x \in S \Rightarrow f \text{ is strictly convex}}$

## Thm 17.13 SemiDefiniteness & Concavity/Convexity

(a) $(-1)^r \Delta_r(x) \geq 0$ : for $r = 1, 2, 3, \ldots, n$ & $\forall x \in S \Leftrightarrow f$ is concave

$\Updownarrow$

$\boxed{H(x) \text{ is N.S.D for } \forall x \in S \Leftrightarrow f \text{ is concave}}$

(b) $\Delta_r(x) \geq 0$ for $r = 1, 2, 3, \ldots, n$ & $\forall x \in S \Leftrightarrow f$ is convex

$\Updownarrow$

$\boxed{H(x) \text{ is P.S.D for } \forall x \in S \Leftrightarrow f \text{ is convex}}$

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